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Techniques for Solving Some Fractional Integrals

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Abstract: **This paper studies some fractional integrals based on Jumarie type of Riemann Liouville (R-L) fractional calculus. The main method used in this article is the change of variables for fractional calculus. A new multiplication of fractional analytic functions plays an important role in this paper. We give some examples to illustrate how to evaluate the fractional integrals. And these results we obtained are natural generalizations of the results in classical calculus.**

Keywords: **Fractional integrals, Jumarie type of R-L fractional calculus, change of variables for fractional calculus, new multiplication, fractional analytic functions.**

I. INTRODUCTION

Fractional calculus originated in 1695, nearly at the same time as conventional calculus. However, in spite of the contributions of important mathematicians, physicists and engineers, fractional calculus still attracted limited attention and remained a pure mathematical exercise. Fractional calculus had a rapid development during the last few decades, both in mathematics and in the applied sciences. Now it is recognized as an excellent tool to describe complex systems, phenomena involving long range memory effects and non-locality. A large number of research papers and books devoted to this subject have been published. At present, the popularity of fractional calculus has attracted many researchers all over the world, and has been widely used in physics, engineering, biology, medicine, economy and finance [1-8].

In this article, based on Jumarie's modification of R-L fractional calculus, we mainly use the change of variables for fractional calculus to solve some fractional integrals. A new multiplication of fractional analytic functions plays an important role in this paper. We provide several examples to illustrate how to evaluate the fractional integrals. In fact, the results we obtained are generalizations of those in traditional calculus.

II. PRELIMINARIES

In the following, the fractional calculus used in this paper and some properties are introduced.

Definition 2.1 ([9])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt. \quad (1)$$

And the Jumarie type of R-L α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt, \quad (2)$$

where $\Gamma(\ \ )$ is the gamma function.

Proposition 2.2 ([10])**:** *Suppose that* α, β, x_0, C *are real numbers and* $\beta \geq \alpha > 0$, *then*

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$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right]=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C]=0 \quad . \tag{4}$$

Next, we introduce the fractional analytic function.

Definition 2.3 ([11])**:** Let x, x_0, and a_k be real numbers for all k, $x_0 \in (a,b)$, and $0<\alpha\le 1$. If the function $f_\alpha:[a,b]\to R$ can be expressed as $f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$, an α-fractional power series on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha:[a,b]\to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, a new multiplication of fractional analytic functions is introduced.

Definition 2.4 ([12])**:** If $0<\alpha\le 1$, and x_0 is a real number. Let $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ be two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}, \tag{5}$$

$$g_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \tag{6}$$

Then we define

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes g_\alpha(x^\alpha) \\ &=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}\otimes\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \\ &=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)(x-x_0)^{k\alpha}. \end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes g_\alpha(x^\alpha) \\ &=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}\otimes\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k} \\ &=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k} \quad . \end{aligned} \tag{8}$$

Definition 2.5 ([13])**:** Let $0<\alpha\le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}, \tag{9}$$

$$g_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \tag{10}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha\circ g_\alpha)(x^\alpha)=f_\alpha\left(g_\alpha(x^\alpha)\right)=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(g_\alpha(x^\alpha)\right)^{\otimes k}, \tag{11}$$

and

$$(g_\alpha\circ f_\alpha)(x^\alpha)=g_\alpha\left(f_\alpha(x^\alpha)\right)=\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(f_\alpha(x^\alpha)\right)^{\otimes k}. \tag{12}$$

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Definition 2.6 ([13])**:** Let $0 < \alpha \leq 1$. If $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions at x_0 satisfies

$$(f_\alpha \circ g_\alpha)(x^\alpha) = (g_\alpha \circ f_\alpha)(x^\alpha) = \frac{1}{\Gamma(\alpha+1)}(x - x_0)^\alpha. \tag{13}$$

Then $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are called inverse functions of each other.

Next, The followings are some fractional analytic functions.

Definition 2.7([14])**:** If $0 < \alpha \leq 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}. \tag{14}$$

And the α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$. In addition, the α-fractional cosine and sine function are defined respectively as follows:

$$cos_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k\alpha}}{\Gamma(2k\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2k}, \tag{15}$$

and

$$sin_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(2k+1)}. \tag{16}$$

On the other hand,

$$sec_\alpha(x^\alpha) = \left(cos_\alpha(x^\alpha)\right)^{\otimes -1} \tag{17}$$

is called the α-fractional secant function.

$$csc_\alpha(x^\alpha) = \left(sin_\alpha(x^\alpha)\right)^{\otimes -1} \tag{18}$$

is the α-fractional cosecant function.

$$tan_\alpha(x^\alpha) = sin_\alpha(x^\alpha) \otimes sec_\alpha(x^\alpha) \tag{19}$$

is the α-fractional tangent function. And

$$cot_\alpha(x^\alpha) = cos_\alpha(x^\alpha) \otimes csc_\alpha(x^\alpha) \tag{20}$$

is the α-fractional cotangent function.

In the following, inverse fractional trigonometric functions are introduced.

Definition 2.8 [15]**:** Let $0 < \alpha \leq 1$. Then $arcsin_\alpha(x^\alpha)$ is the inverse function of $sin_\alpha(x^\alpha)$, and it is called inverse α-fractional sine function. $arccos_\alpha(x^\alpha)$ is the inverse function of $cos_\alpha(x^\alpha)$, and we say that it is the inverse α-fractional cosine function. On the other hand, $arctan_\alpha(x^\alpha)$ is the inverse function of $tan_\alpha(x^\alpha)$, and it is called the inverse α-fractional tangent function. $arccotan_\alpha(x^\alpha)$ is the inverse function of $cot_\alpha(x^\alpha)$, and we say that it is the inverse α-fractional cotangent function. $arcsec_\alpha(x^\alpha)$ is the inverse function of $sec_\alpha(x^\alpha)$, and it is the inverse α-fractional secant function. $arccsc_\alpha(x^\alpha)$ is the inverse function of $csc_\alpha(x^\alpha)$, and is called the inverse α-fractional cosecant function.

Definition 2.9 [16]**:** Let $0 < \alpha \leq 1$, and r be a real number. The r-th power of the α-fractional analytic function $f_\alpha(x^\alpha)$ is defined by $[f_\alpha(x^\alpha)]^{\otimes r} = E_\alpha\left(r \cdot Ln_\alpha\left(f_\alpha(x^\alpha)\right)\right)$.

III. TECHNIQUES AND EXAMPLES

In this section, some methods used in this paper are introduced, and we provide several examples to illustrate how to evaluate some fractional integrals.

Theorem 3.1 (change of variables for fractional calculus)[17]**:** *If $0 < \alpha \leq 1$, $t_\alpha(x^\alpha)$ is an α-fractional analytic function defined on an interval I, and $f_\alpha\left(t_\alpha(x^\alpha)\right)$ is an α-fractional analytic function such that the range of t_α contained in the domain of f_α, then*

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$$\left({}_{t_\alpha(c^\alpha)}I^\alpha_{t_\alpha(d^\alpha)}\right)[f_\alpha(t_\alpha)] = \left({}_cI^\alpha_d\right)\left[f_\alpha\left(t_\alpha(x^\alpha)\right) \otimes \left({}_cD^\alpha_x\right)[t_\alpha(x^\alpha)]\right], \tag{21}$$

for $c, d \in I$.

Theorem 3.2 [15]**:** *Let* $0 < \alpha \le 1$, $x \ge 0$ *, and* x *be a real number. Then*

$$\left({}_0I^\alpha_x\right)\left[\left(1 + \frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes -1}\right] = Ln_\alpha\left(1 + \frac{1}{\Gamma(\alpha+1)}x^\alpha\right), \tag{22}$$

$$\left({}_0I^\alpha_x\right)\left[\left(1 + \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)^{\otimes -1}\right] = arctan_\alpha(x^\alpha). \tag{23}$$

Theorem 3.3 ([18])**:** *If* $0 < \alpha \le 1$, *and* x *is a real number, then*

$$[sin_\alpha(x^\alpha)]^{\otimes 2} + [cos_\alpha(x^\alpha)]^{\otimes 2} = 1, \tag{24}$$

$$1 + [tan_\alpha(x^\alpha)]^{\otimes 2} = [sec_\alpha(x^\alpha)]^{\otimes 2}, \tag{25}$$

$$1 + [cot_\alpha(x^\alpha)]^{\otimes 2} = [csc_\alpha(x^\alpha)]^{\otimes 2}. \tag{26}$$

Example 3.4: If $0 < \alpha \le 1$, and $t \ge p \ge \left(\Gamma(\alpha+1)\right)^{\frac{1}{\alpha}}$. Evaluate the α-fractional integral

$$\left({}_pI^\alpha_t\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes -1} \otimes \left(\frac{1}{\Gamma(\alpha+1)}t^\alpha - 1\right)^{\otimes\left(\frac{1}{2}\right)}\right]. \tag{27}$$

Solution Let $\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha - 1\right)^{\otimes\left(\frac{1}{2}\right)} = \frac{1}{\Gamma(\alpha+1)}x^\alpha$, then $\frac{1}{\Gamma(\alpha+1)}t^\alpha = \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2} + 1$. Let $\left(\frac{1}{\Gamma(\alpha+1)}p^\alpha - 1\right)^{\otimes\left(\frac{1}{2}\right)} = \frac{1}{\Gamma(\alpha+1)}s^\alpha$, by change of variables for fractional calculus, we obtain

$$\left({}_pI^\alpha_t\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes -1} \otimes \left(\frac{1}{\Gamma(\alpha+1)}t^\alpha - 1\right)^{\otimes\left(\frac{1}{2}\right)}\right]$$

$$= \left({}_sI^\alpha_x\right)\left[\left(\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2} + 1\right)^{\otimes -1} \otimes 2 \cdot \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right]$$

$$= 2 \cdot \left({}_sI^\alpha_x\right)\left[1 - \left(\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2} + 1\right)^{\otimes -1}\right]$$

$$= 2 \cdot \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha - arccotan_\alpha(x^\alpha)\right) - 2 \cdot \left(\frac{1}{\Gamma(\alpha+1)}s^\alpha - arccotan_\alpha(s^\alpha)\right)$$

$$= 2 \cdot \left(\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha - 1\right)^{\otimes\left(\frac{1}{2}\right)} - arccotan_\alpha\left(\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha - 1\right)^{\otimes\left(\frac{1}{2}\right)}\right)\right)$$

$$-2 \cdot \left(\left(\frac{1}{\Gamma(\alpha+1)}p^\alpha - 1\right)^{\otimes\left(\frac{1}{2}\right)} - arccotan_\alpha\left(\left(\frac{1}{\Gamma(\alpha+1)}p^\alpha - 1\right)^{\otimes\left(\frac{1}{2}\right)}\right)\right). \tag{28}$$

Example 3.5: Let $0 < \alpha \le 1$, and $t \ge 0$. Find the α-fractional integral

$$\left({}_0I^\alpha_t\right)\left[\left(1 + \left(\frac{1}{\Gamma(\alpha+1)}t^\alpha + 2\right)^{\otimes\frac{1}{3}}\right)^{\otimes -1}\right]. \tag{29}$$

Solution Let $\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha + 2\right)^{\otimes\frac{1}{3}} = \frac{1}{\Gamma(\alpha+1)}x^\alpha$, then $\frac{1}{\Gamma(\alpha+1)}t^\alpha = \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 3} - 2$. Let $q = \frac{1}{1+\sqrt[3]{2}}$, also using change of variables for fractional calculus, we have

$$\left({}_{0}I_{t}^{\alpha}\right)\left[\left(1+\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}+2\right)^{\otimes\frac{1}{3}}\right)^{\otimes -1}\right]$$

$$=\left({}_{q}I_{x}^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)^{\otimes -1}\otimes 3\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right]$$

$$=3\cdot\left({}_{q}I_{x}^{\alpha}\right)\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)^{\otimes -1}\right]$$

$$=3\cdot\left[\frac{1}{2}\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+Ln_{\alpha}\left(1+\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\right]$$

$$-3\cdot\left[\frac{1}{\Gamma(2\alpha+1)}q^{2\alpha}-\frac{1}{\Gamma(\alpha+1)}q^{\alpha}+Ln_{\alpha}\left(1+\frac{1}{\Gamma(\alpha+1)}q^{\alpha}\right)\right]$$

$$=\frac{3}{2}\cdot\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}+2\right)^{\otimes\frac{2}{3}}-3\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}+2\right)^{\otimes\frac{1}{3}}+3\cdot Ln_{\alpha}\left(1+\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}+2\right)^{\otimes\frac{1}{3}}\right)$$

$$-\left(\frac{3}{2}\cdot\sqrt[3]{4}-3\cdot\sqrt[3]{2}+3\cdot Ln_{\alpha}\left(1+\sqrt[3]{2}\right)\right). \quad (30)$$

Example 3.6: Suppose that $0<\alpha\leq 1, p>0,\ sin_{\alpha}(p^{\alpha})\neq 0$, and $t\geq p$. Evaluate the α-fractional integral

$$\left({}_{p}I_{t}^{\alpha}\right)\left[\left(1+sin_{\alpha}(t^{\alpha})\right)\otimes\left[sin_{\alpha}(t^{\alpha})\otimes\left(cos_{\alpha}(t^{\alpha})+1\right)\right]^{\otimes -1}\right]. \quad (31)$$

Solution Since

$$sin_{\alpha}(t^{\alpha})=2\cdot sin_{\alpha}\left(\frac{1}{2}t^{\alpha}\right)\otimes cos_{\alpha}\left(\frac{1}{2}t^{\alpha}\right)$$

$$=2\cdot tan_{\alpha}\left(\frac{1}{2}t^{\alpha}\right)\otimes\left(\left(sec_{\alpha}\left(\frac{1}{2}t^{\alpha}\right)\right)^{\otimes 2}\right)^{\otimes -1}$$

$$=2\cdot tan_{\alpha}\left(\frac{1}{2}t^{\alpha}\right)\otimes\left(1+\left(tan_{\alpha}\left(\frac{1}{2}t^{\alpha}\right)\right)^{\otimes 2}\right)^{\otimes -1}. \quad (32)$$

And

$$cos_{\alpha}(t^{\alpha})=\left(cos_{\alpha}\left(\frac{1}{2}t^{\alpha}\right)\right)^{\otimes 2}-\left(sin_{\alpha}\left(\frac{1}{2}t^{\alpha}\right)\right)^{\otimes 2}$$

$$=\left[1-\left(tan_{\alpha}\left(\frac{1}{2}t^{\alpha}\right)\right)^{\otimes 2}\right]\otimes\left(\left(sec_{\alpha}\left(\frac{1}{2}t^{\alpha}\right)\right)^{\otimes 2}\right)^{\otimes -1}$$

$$=\left[1-\left(tan_{\alpha}\left(\frac{1}{2}t^{\alpha}\right)\right)^{\otimes 2}\right]\otimes\left(1+\left(tan_{\alpha}\left(\frac{1}{2}t^{\alpha}\right)\right)^{\otimes 2}\right)^{\otimes -1}. \quad (33)$$

Let $\frac{1}{\Gamma(\alpha+1)}x^{\alpha}=tan_{\alpha}\left(\frac{1}{2}t^{\alpha}\right)$, then $\frac{1}{\Gamma(\alpha+1)}t^{\alpha}=2\cdot arctan_{\alpha}(x^{\alpha})$, and

$$sin_{\alpha}(t^{\alpha})=\left[2\cdot\ \frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]\otimes\left(1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right)^{\otimes -1}, \quad (34)$$

$$cos_{\alpha}(t^{\alpha})=\left[1-\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right]\otimes\left(1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right)^{\otimes -1}. \quad (35)$$

Let $q=\left[\Gamma(\alpha+1)\cdot tan_{\alpha}\left(\frac{1}{2}p^{\alpha}\right)\right]^{\frac{1}{\alpha}}$, then

$$\left({}_{q}I_{x}^{\alpha}\right)\left[2\cdot arctan_{\alpha}(x^{\alpha})\right]=2\cdot\left(1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right)^{\otimes -1}-2\cdot\left(1+\left(\frac{1}{\Gamma(\alpha+1)}q^{\alpha}\right)^{\otimes 2}\right)^{\otimes -1}. \quad (36)$$

Therefore,

$$\left({}_{p}I_{t}^{\alpha}\right)\left[\left(1+sin_{\alpha}(t^{\alpha})\right)\otimes\left[sin_{\alpha}(t^{\alpha})\otimes\left(cos_{\alpha}(t^{\alpha})+1\right)\right]^{\otimes -1}\right]$$

$$=\left({}_{q}I_{x}^{\alpha}\right)\begin{bmatrix}\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}+2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]\otimes\left(1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right)^{\otimes -1}\otimes\left(2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes -1}\\ \otimes\left(1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right)\otimes\frac{1}{2}\left(1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right)\otimes 2\cdot\left(1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right)^{\otimes -1}\end{bmatrix}$$

$$=\frac{1}{2}\left({}_{q}I_{x}^{\alpha}\right)\left[\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}+2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]\otimes\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes -1}\right]$$

$$=\frac{1}{2}\left({}_{q}I_{x}^{\alpha}\right)\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+2+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes -1}\right]$$

$$=\frac{1}{2}\left[\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha}+2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+Ln_{\alpha}(x^{\alpha})\right]-\frac{1}{2}\left[\frac{1}{\Gamma(2\alpha+1)}q^{2\alpha}+2\cdot\frac{1}{\Gamma(\alpha+1)}q^{\alpha}+Ln_{\alpha}(q^{\alpha})\right]$$

$$=\left[\frac{1}{4}\left(tan_{\alpha}\left(\frac{1}{2}t^{\alpha}\right)\right)^{\otimes 2}+tan_{\alpha}\left(\frac{1}{2}t^{\alpha}\right)+\frac{1}{2}Ln_{\alpha}\left(\left|tan_{\alpha}\left(\frac{1}{2}t^{\alpha}\right)\right|\right)\right]$$

$$-\left[\frac{1}{4}\left(tan_{\alpha}\left(\frac{1}{2}p^{\alpha}\right)\right)^{\otimes 2}+tan_{\alpha}\left(\frac{1}{2}p^{\alpha}\right)+\frac{1}{2}Ln_{\alpha}\left(\left|tan_{\alpha}\left(\frac{1}{2}p^{\alpha}\right)\right|\right)\right]. \quad (37)$$

IV. CONCLUSION

As mentioned above, based on Jumarie type of R-L fractional calculus and a new multiplication, this paper studies how to solve several fractional integrals. The main method we used is the change of variables for fractional calculus. In addition, the results obtained in this article are natural generalizations of those in traditional calculus. Furthermore, the new multiplication we defined is a natural operation of fractional analytic functions. In the future, we will use the new multiplication to solve some problems in fractional differential equations and applied mathematics.

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